The structure of stable constant mean curvature hypersufaces

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Abstract

We study the global behavior of (weakly) stable constant mean curvature hypersurfaces in general Riemannian manifolds. By using harmonic function theory, we prove some one-end theorems which are new even for constant mean curvature hypersurfaces in space forms. In particular, a complete oriented weakly stable minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, must have only one end. Any complete noncompact weakly stable CMC H-hypersurface in the hyperbolic space $\mathbb{H}^{n+1}, n = 3, 4$, with $H^2 \geq \frac{10}{9}, \frac{7}{4}$, respectively, has only one end.

0 Introduction

The classical Bernstein theorem states that a minimal entire graph in $\mathbb{R}^3$ must be planar. This theorem was later generalized to higher dimensions (dimension of the ambient Euclidean space $\mathbb{R}^{n+1}$ is no more than 8) by Fleming[Fl], Almgren[A], De Giorgi[Dg], and Simons[S]. In $\mathbb{R}^{n+1}$, $n \geq 8$, the examples of nonlinear entire graphs are given by Bombieri, de Giorgi and Giusti [BdGG]. Because of the stability of minimal entire graphs, one is naturally led to the generalization of the classical Bernstein theorem to the question of asking whether all stable minimal hypersurfaces in R^{n+1} are hyperplanes when $n \leq 7$. In the case when $n = 2$, this problem was solved independently by do Carmo and Peng [dCP]; and Fischer-Colbrie and Schoen [FS]. For higher

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dimension, this problem is still open. On the other hand, there are some results about the structure of stable minimal hypersurfaces in all $\mathbb{R}^{n+1}$. For instance, H. Cao, Y. Shen and S. Zhu [CSZ] proved that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, must have only one end.

If the ambient manifold is not the Euclidean space, Fischer-Colbrie and Schoen [FS] gave a classification for complete oriented stable minimal surfaces in a complete oriented 3-manifold of nonnegative scalar curvature. Recently, Li and Wang [LW1] showed that a complete noncompact properly immersed stable minimal hypersurface in a complete manifold of nonnegative sectional curvature must either have only one end or be totally geodesic and a product of a compact manifold with nonnegative sectional curvature and $\mathbb{R}$.

In this paper we study hypersurfaces with constant mean curvature H. Let us now fix terminologies and notations so as to our theorems. In the sequel we will abbreviate constant mean curvature hypersurfces by calling them CMC $H-$hypersurfaces and will allow H to vanish (hence the need of putting H here). Instead of the usual stability, we will consider a weaken form of stability, which is in fact the natural one for CMC $H-$hypersurfaces in case $H \neq 0$. Intuitively, a CMC hypersurface is weakly stable if the second variations are nonnegative for all compactly supported enclosed-volume-preserving variations (see Definition 1.1 and Remark 1.1). This concept of weakly stable CMC hypersurfaces was introduced by Barbosa, do Carmo and Eschenburg [BdCE], to accounts for the fact that spheres are stable (see [BdCE]). This weak stability comes naturally from the phenomenon of soap bubbles and is related to isoperimetric problems. In [dS], da Silveira studied complete noncompact weakly stable CMC surfaces in $\mathbb{R}^3$ or the hyperbolic space $\mathbb{H}^3$. He proved that complete weakly stable CMC surfaces in $\mathbb{R}^3$ are planes and hence generalized the corresponding result of do Carmo and Peng [dCP], Fischer-Colbrie and Schoen [FS]. For $\mathbb{H}^3$ he shows that only horospheres can occur when constant mean curvature $|H| \geq 1$. For higher dimensions, very little is known about complete noncompact weakly stable CMC hypersurfaces.

In this paper, we study the global behavior of weakly stable CMC hypersurface (including minimal case). First, we obtain

Theorem 0.1. *(Th.3.4) Let $N^{n+1}, n \geq 5$, be a complete Riemannian manifold and M be a complete noncompact weakly stable immersed CMC H-hypersurface in N. If one of the following cases occurs,*

(1) when $n = 5$, the sectional curvature of N is nonnegative and $H \neq 0$;

(2) when $n \geq 6$, the sectional curvature $\tilde{K}$ of N satisfies $\tilde{K} \geq \tau > 0$ and $H^2 \leq \frac{4(2n-1)}{n^2(n-5)}\tau$, for some number $\tau > o$;

(3) when $n \geq 6$, the sectional curvature and the Ricci curvature of N satisfy $\tilde{K} \geq 0$, $\tilde{\mathrm{Ric}} \geq \tau > 0$, for some number $\tau > 0$, and $H = 0$,

then M has only one end.

The reason for the restriction on dimensions of CMC hypersurfaces in the above theorem is that there are some nonexistence results (see the proof of this theorem for detail). Theorem 0.1 has the following examples: complete noncompact weakly stable CMC H-hypersurfaces in the standard sphere $\mathbb{S}^6$ with $H \neq 0$; or in the standard sphere $\mathbb{S}^{n+1}, n \geq 6$ with $H^2 \leq \frac{4(2n-1)}{n^2(n-5)}$.

Actually, Theorem 0.1 is a special case of the more general Theorem 3.3, which also implies that

Theorem 0.2. *(Cor.3.3) Any complete noncompact weakly stable CMC H-hypersurface in the hyperbolic space $\mathbb{H}^{n+1}, n = 3, 4$, with $H^2 \geq \frac{10}{9}, \frac{7}{4}$, respectively, has only one end.*

Next we consider complete weakly stable minimal hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$, and generalize the results of Cao, Shen and Zhu as follows:

Theorem 0.3. *(Th.3.2) A complete oriented weakly stable minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, must have only one end.*

With this theorem, we obtain

Corollary 0.1. *(Cor.3.1) A complete oriented weakly stable immersed minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with finite total curvature (i.e., $\int_M |A|^n < \infty$) is a hyperplane.*

Finally, we study the structure of weakly stable CMC hypersurfaces according to the parabolicity or nonparabolicity of M. We obtain the following results:

Theorem 0.4. *(Th.5.2) Let N be a complete manifold of bounded geometry and M be a complete noncompact weakly stable CMC H-hypersurface immersed in N. If the sectional curvature of N is bounded from below by $-H^2$ and M is parabolic, then it is totally umbilic and has nonnegative sectional curvature. Furthermore, either*

(1) M has only one end; or

(2) $M = \mathbb{R} \times P$ with the product metric, where P is a compact manifold of nonnegative sectional curvature.

Theorem 0.5. *(Th.5.3) Let N be a complete Riemannian manifold and M be a complete noncompact weakly stable CMC H-hypersurface immersed in N. If M is nonparabolic, and*

$$\tilde{\mathrm{Ric}}(\nu) + \tilde{\mathrm{Ric}}(X) - \tilde{\mathrm{K}}(X,\nu) \geq \frac{n^2(n-5)}{4} H^2, \forall X \in T_pM, |X| = 1, p \in M,$$

then it has only one nonparabolic end, where $\tilde{\mathrm{K}}$ and $\tilde{\mathrm{Ric}}$ denote the sectional and Ricci curvatures of N, respectively; ν denotes the unit normal vector field of M.

In some of recent works, the structure of stable (i.e., strongly stable) minimal hypersurfaces was studied by means of harmonic function theory (see [CSZ], [LW], [LW1]). The same approach can be used in the study of weakly stable CMC hypersurfaces. However, a significant difference between weakly stable and strongly stable cases lies in the choice of test functions. When one deals with weak stability, the test functions f must satisfy $\int_M f = 0$. In this paper, we successfully construct the required test functions by using the properties of harmonic functions (Theorem 3.1 and Proposition 4.1). Combining our construction and the approach in [LW], [LW1], we are able to discuss the global behavior of weakly stable CMC hypersurfaces. In Theorem 3.1, we obtain the nonexistence of nonconstant bounded harmonic functions with finite Dirichlet integral on weakly stable CMC hypersurfaces. This theorem enable us to study the uniqueness of ends. In Proposition 4.1, we discuss a property of Schrödinger operator on parabolic manifolds which can be applied to study weakly stable CMC hypersurfaces with parabolicity. Besides, different from minimal hypersurfaces, CMC hypersurfaces with $H \neq 0$ have the curvature estimate depending on H, which causes dimension restriction in the results.

The rest of this paper is organized as follows: in Section 1 we give some definitions and facts as preliminaries; in Section 2, we first discuss volume growth of the ends of complete noncompact hypersurfaces with mean curvature vector field bounded in norm, and then study nonparabolicity of the ends of CMC hypersurfaces with stability; in Section 3, we use harmonic functions to study the uniqueness of ends of complete noncompact weakly stable CMC hypersurfaces; in Section 4, we give a property of Schrödinger operator on parabolic manifolds; in the last section (Section 5), we discuss the structure of complete noncompact weakly stable CMC hypersurfaces.

The results on minimal case in this paper has been announced in [CCZ].

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1 Preliminaries

We recall some definitions and facts in this section.

Let N^{n+1} be an oriented $(n+1)$-dimensional Riemannian manifold and let $i : M^n \to N^{n+1}$ be an isometric immersion of a connected n-dimensional manifold M with constant mean curvature H. We assume M is orientable. When H is nonzero, the orientation is automatic. Throughout this paper, $\tilde{\mathrm{K}}, \tilde{\mathrm{Ric}}$, K, and Ric denote the sectional, Ricci curvatures of N, the sectional, Ricci curvature of M respectively. ν denotes the unit normal vector field of M. $|A|$ is the norm of the second fundamental form A. $B_p(R)$ will denote the intrinsic geodesic ball in M of radius R centered at p. We have

Definition 1.1. *There are two cases. In the case $H \neq 0$, the immersion i is called stable or weakly stable if*

$$\int_M \{|\nabla f|^2 - \left(\tilde{\mathrm{Ric}}(\nu,\nu) + |A|^2\right) f^2\} \geq 0, \tag{1.1}$$

for all compactly supported piecewise smooth functions $f\colon M \to R$ satisfying

$$\int_M f = 0,$$

where ∇f is the gradient of f in the induced metric of M;

the immersion i is called strongly stable if (1.1) holds for all compactly supported piecewise smooth functions $f\colon M \to R$.

In the case $H = 0$ (minimal case), the immersion i is called weakly stable if (1.1) holds for all compactly supported piecewise smooth functions satisfying $f\colon M \to R$

$$\int_M f = 0;$$

the immersion i is called stable if (1.1) holds for all compactly supported piecewise smooth functions $f\colon M \to R$.

It is known, from the definition, that a weakly stable minimal hypersurface has the index 0 or 1 (see [dS]). Obviously, a strongly stable CMC hypersurface is weakly stable. But the converse may not be true. For example, $\mathbb{S}^2 \subset \mathbb{S}^3$ as a totally geodesic embedding in the ordinary 3-sphere is not stable but weakly stable.

Remark 1.1. *In the current literatures, the terms of stability on minimal and constant mean curvature hypersurfaces are different (perhaps a little confusing). A hypersurface with nonzero constant mean curvature is called stable if it is weakly stable; while a minimal hypersurface is called stable if it is strongly stable in the above sense. In this paper, we deal with the weak stability for both hypersurfaces. In order to avoid confusion and conform to the notations of others, the notation of weak stability is used without omission in this paper.*

For CMC H-hypersurfaces, it is convenient to introduce the (traceless) tensor $\Phi := A - HI$, where I denotes the identity. A straightforward computation gives $|\Phi|^2 = |A|^2 - nH^2$ and the stability inequality (1.1) becomes

$$\int_M \{|\nabla f|^2 - \left(\tilde{\mathrm{Ric}}(\nu, \nu) + |\Phi|^2 + nH^2\right) f^2\} \geq 0. \tag{1.2}$$

In this paper, we will discuss the number of ends of hypersurfaces. Now we give some related definitions.

Definition 1.2. *(cf. [LT], [LW]) A manifold is said to be parabolic if it does not admit a positive Green's function. Conversely, a nonparabolic manifold is one which admits a positive Green's function.*

An end E of Σ is said to be nonparabolic if it admits a positive Greens function with Neumann boundary condition on ∂E. Otherwise, it is said to be parabolic.

In order to estimate the number of ends of a weakly stable CMC hypersurface, we need the following theorem by Li and Tam.

Theorem 1.1. *([LT], see also[LW] Theorem 1) Let M be a complete Riemannian manifold. Let $\mathcal{H}^0_D(M)$ be the space of bounded harmonic functions with finite Dirichlet integral. Then the number of nonparabolic ends of M is bounded from above by $dim\mathcal{H}^0_D(M)$.*

From Theorem 1.1, we know that if every end of M is nonparabolic, then the number of its ends is no more than $\dim \mathcal{H}^0_D(M)$.

2 Nonparabolicity of ends

In this section, we first discuss the volume growth of ends of complete noncompact submanifolds in a Riemannian manifold N of bounded geometry (*a manifold N is called bounded geometry if its sectional curvatures $\tilde{\mathrm{K}} \leq \sigma^2, \sigma > 0$ and its injectivity radius $i_N(p) \geq i_0, i_0 > 0$*) and using it to study the property of the nonparabolic ends of submanifolds.

Frensel [Fr] showed that if M is a complete noncompact immersed submanifold in a manifold of bounded geometry with mean curvature vector field bounded in norm, then M has infinite volume. Here, we prove that even each end of M has infinite volume.

Lemma 2.1. *([Fr] Th.3) Let N be an m-dimensional manifold and let M be an n-dimensional complete noncompact manifold. Let $x : M^n \to N^m$ be an isometric immersion with mean curvature vector field bounded in norm. Assume that N has sectional curvature $\tilde{\mathrm{K}} \leq \sigma^2$, where constant $\sigma > 0$. Then*

$$\mathrm{Vol}(B_p(R)) \geq \sigma^{-n}\omega_n(\sin R\sigma)^n e^{-H_0 R},$$

where $R \leq \min\{\frac{\pi}{2\sigma}, i_N(p)\}$ and $|H| \leq H_0$.

We obtain

Proposition 2.1. *Let N be an m-dimensional manifold of bounded geometry and let M be an n-dimensional complete noncompact manifold. Let $x : M \to N$ be an isometric immersion with mean curvature vector field bounded in norm. Then each end E of M has infinite volume. More exactly, the rate of volume growth of E is at least linear, i.e., for any $p \in E$,*

$$\liminf_{R\to\infty} \frac{\mathrm{Vol}(B_p(R) \cap E)}{R} > 0, \tag{2.1}$$

where the limit is independent of the choice of p.

Proof. Assume that E is an end of M with respect to a compact set $D \subset M$ with smooth boundary ∂D.

We claim that there exist some $x \in E$ and a ray γ in E emanating from x, i.e., $\gamma : [0, \infty) \to E$ is a minimizing geodesic satisfying $\gamma(0) = x$, and $d(\gamma(s), \gamma(t)) = |s - t|$, for all $s, t \geq 0$, where γ has the arc length parameter.

Now we prove the claim. Since E is unbounded, there exists a sequence of points $q_i \in E$ such that $d(q_i, D) \to \infty$ when $i \to \infty$. Since D is compact, there exist a sequence of points $p_i \in \partial D$ and a sequence of minimizing

normalized geodesic segments $\gamma_i|_{[0,s_i]}$ in M joining $p_i = \gamma_i(0)$ to $q_i = \gamma(s_i)$ respectively, such that $d(q_i, p_i) = d(q_i, D)$.

Each γ_i has the following properties: 1) p_i is the only intersection of D and γ_i (otherwise, $d(q_i, p_i) \neq d(q_i, D)$); 2) $\gamma_i \backslash \{p_i\} \subset E$ (since E is a connected component of $M \backslash D$); 3) $\gamma_i'(0)$ is orthogonal to D at p_i (since γ_i realizes the distance $d(q_i, D)$).

Since the unit normal bundle of D is compact, there exists a subsequence of $(p_i, \gamma_i'(0))$, which is still denoted by $(p_i, \gamma_i'(0))$, converging to a point (p_0, ν) in the unit normal bundle, where $p_0 \in D, \nu \in T_{p_0}M$. Let $\tilde{\gamma}|_{[0,+\infty)}$ be the normalized geodesic in M emanating from p_0 with initial unit tangent vector ν. By ODE theory, γ_i converges to $\tilde{\gamma}$ uniformly on any compact subset of $[0, +\infty)$. Moreover, for any $s \in [0, +\infty)$, the segment $\tilde{\gamma}|_{[0,s]}$ realizes the distance from $\tilde{\gamma}(s)$ to D. By the same reason, $\tilde{\gamma}$ also has the properties $1)-3)$ like γ_i.

Choose $x = \tilde{\gamma}(a) \in \tilde{\gamma} \backslash \{p_0\}$ and take $\gamma(s) = \tilde{\gamma}(a + s), s \geq 0$. We obtain a ray γ in E emanating from $x \in E$ as claimed.

Note for any $z \in \gamma$, $d(z, D) \geq a > 0$. So we may choose small R_0 ($R_0 < a$) such that $B_z(R_0) \subset E$, $z \in \gamma$. Take R_0 satisfying $R_0 < \min\{\frac{\pi}{2\sigma}, i_0, a\}$. By Lemma 2.1, for any $z \in \gamma \subset M$,

$$\mathrm{Vol}(B_z(R_0) \geq \sigma^{-n}\omega_n(\sin R_0\sigma)^n e^{-H_0R_0} = \beta > 0. \tag{2.2}$$

Consider a sequence of points $z_j = \gamma(2jR_0), j = 0, ..., k-1,$ where $k = \left[\frac{R}{2R_0}\right]$, $R \geq 2R_0$. Observe that any two balls $B_{z_j}(R_0)$ are disjoint and $B_x(R) \supset \bigcup_{j=0}^{k-1} B_{z_j}(R_0)$. Then, $B_x(R) \cap E \supset \bigcup_{j=0}^{k-1} B_{z_j}(R_0)$, and by (2.2),

$$\mathrm{Vol}(B_x(R) \cap E) \geq \mathrm{Vol}(\bigcup_0^{k-1} B_{z_j}(R_0)) \geq k\beta \geq (\frac{R}{2R_0} - 1)\beta, \quad R \geq 2R_0.$$

Hence

$$\liminf_{R\to\infty} \frac{\mathrm{Vol}(B_x(R) \cap E)}{R} > 0.$$

It is direct, from the definition of lim inf, that limit is independent of the choice of p and hence constant for any point of E.

$\square$

Corollary 2.1. *Let N be a complete simply connected manifold of nonpositive sectional curvature and M be a complete noncompact immersed submanifold in N^m with norm-bounded mean curvature vector field H. Then each end of M has infinite volume.*

Li and Wang ([LW], Corollary 4) showed that if an end of a manifold is of infinite volume and satisfies a Sobolev type inequality, then this end must be nonparabolic. With this property, we obtain Proposition 2.2 and Proposition 2.3 as follows:

Proposition 2.2. *Let N^{n+1} be a complete Riemannian manifold of bounded geometry and M^n be a complete noncompact immersed CMC hypersurface in N with finite Morse index. If $\inf \tilde{Ric} > -nH^2$, then each end of M must be nonparabolic.*

Proof. It is well known that a CMC hypersurface with finite Morse index is strongly stable outside a compact domain (by the same argument in [Fc]). Hence we assume that M is stable outside a compact domain $\Omega \subset M$. Clearly each end of M is also stable outside Ω.

Since nonparabolicity of an end depends only on its infinity behavior, it is sufficient to show that each end E of M with respect to any compact set D ($\Omega \supset D$) is nonparabolic.

By stability, for any compactly supported function $f \in H_{1,2}(E)$, we have

$$\int_E |\nabla f|^2 \geq \int_E (\tilde{\mathrm{Ric}}(\nu,\nu) + |\Phi|^2 + nH^2) f^2 \geq (\inf \tilde{\mathrm{Ric}} + nH^2) \int_E f^2,$$

that is, the end E satisfies an Sobolev type inequality:

$$\int_E f^2 \leq C \int_E |\nabla f|^2.$$

By Corollary 4 in [LW] and Prop.2.1, E must be nonparabolic. □

Proposition 2.3. *Let N^m be a complete simply connected manifold with nonpositive sectional curvature and let M^n be a complete immersed minimal submanifold in N^m. If $n \geq 3$, then each end of M must be nonparabolic.*

Proof. From the theorem of Cartan-Hadamard, the exponential map at any point of N must be diffeomorphic R^m and hence N has bounded geometry. Assume E is an end of M. Since under the hypotheses of proposition, we have the following Sobolev inequality ([HS], Theorem2.1):

$$\left(\int_E |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C \int_E |\nabla f|^2, f \in H_{1,2}(E). \tag{2.3}$$

By Corollary 4 in [LW] and Proposition 2.1, E must be nonparabolic. □

Remark 2.1. *The special case of Corollary 2.1 and Proposition 2.3 that E is an end of a minimal submanifold in $\mathbb{R}^m$ was proved in [CSZ].*

3 Uniqueness of ends

In this section we discuss the uniqueness of ends of weakly stable CMC hypersurfaces. We initially prove an algebra inequality.

Lemma 3.1. *Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with trace $tr(A) = nH$. Then*

$$nHa_{11} - \sum_{i=1}^{n} a_{1i}^2 \geq (n-1)H^2 - (n-2)|H||B|\sqrt{\frac{n-1}{n}} - \frac{n-1}{n}|B|^2, \quad (3.1)$$

where $B = (b_{ij}) = A - HI, |B|^2 = \sum_{i,j=1}^{n} {b_{ij}}^2$, I is the identity matrix.

Proof. Note $\sum_{i=1}^{n} b_{ii} = 0$. We have

$$b_{11}^2 = \left(\sum_{i=2}^{n} b_{ii}\right)^2 \leq (n-1)\sum_{i=2}^{n} b_{ii}^2.$$

Then

$$\begin{aligned}|B|^2 = \sum_{i,j=1}^{n} {b_{ij}}^2 &\geq b_{11}^2 + \sum_{i=2}^{n} b_{ii}^2 + 2\sum_{i=2}^{n} b_{1i}^2 \\ &\geq b_{11}^2 + \frac{1}{n-1}\left(\sum_{i=2}^{n} b_{ii}\right)^2 + 2\sum_{i=2}^{n} b_{1i}^2 \\ &\geq \frac{n}{n-1}\left(b_{11}^2 + \sum_{i=2}^{n} b_{1i}^2\right).\end{aligned}$$

By $b_{ii} = a_{ii} - H, i = 1, ..., n$; $b_{ij} = a_{ij}$, $i \neq j, i, j = 1, ..., n$, we have

$$\begin{aligned}nHa_{11} - \sum_{i=1}^{n} a_{1i}^2 &= (n-1)H^2 + (n-2)Hb_{11} - (b_{11}^2 + \sum_{i=2}^{n} b_{1i}^2) \\ &\geq (n-1)H^2 - (n-2)|H||B|\sqrt{\frac{n-1}{n}} - \frac{n-1}{n}|B|^2. \quad (3.2)\end{aligned}$$

□

As a consequence, we obtain the following inequality, which was proved in [Ch] (Lemma 2.1 in [Ch]) by a different proof.

Proposition 3.1. *Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with trace $tr(A) = nH$. Then*

$$|A|^2 + nHa_{11} - \sum_{i=1}^{n} a_{1i}^2 \geq \frac{n^2(5-n)}{4} H^2, \tag{3.3}$$

where $|A|^2 = \sum_{i,j=1}^{n} a_{ij}^2$. Moreover equality holds if and only if one of the following cases occurs:

1) $n = 2$, $A = HI$, where I is the identity matrix;

2) $n \geq 3$, A is a diagonal matrix with $a_{11} = -\frac{n(n-1)}{2}H$, $a_{ii} = \frac{n}{2}H, i = 2, ..., n$, and $a_{ij} = 0, i \neq j, i, j = 1, ..., n$.

Proof. We use the same notations in Lemma 3.1. By $|B|^2 = |A|^2 - nH^2$,

$$\begin{aligned}
&|A|^2 + nHa_{11} - \sum_{i=1}^{n} a_{1i}^2 \\
&\geq |B|^2 + (2n-1)H^2 - (n-2)|H||B|\sqrt{\frac{n-1}{n}} - \frac{n-1}{n}|B|^2 \\
&= (\frac{|B|}{\sqrt{n}} - \frac{(n-2)\sqrt{n-1}}{2}|H|)^2 + \frac{n^2(5-n)H^2}{4} \\
&\geq \frac{n^2(5-n)H^2}{4}.
\end{aligned}$$

Thus, we obtain (3.3). If the equality in (3.3) holds, then,

1) in the case $n = 2$, $\frac{|B|}{\sqrt{n}} - \frac{(n-2)\sqrt{n-1}}{2}|H| = 0$, so $B = 0$, that is, $A = HI$.

2) in the case $n \geq 3$, by the proof of (3.3), we have $\sum_{i=1}^{n} b_{ii} = 0$; $b_{ii} = b_{jj}, i, j = 2, ..., n$; $b_{ij} = 0$, $i \neq j, i, j = 1, ..., n$; $\frac{|B|}{\sqrt{n}} - \frac{(n-2)\sqrt{n-1}}{2}|H| = 0$. Moreover, b_{11} and H have different signs.

Thus, $b_{11} = -(n-1)b_{22}, |b_{11}| = \frac{\sqrt{n-1}}{\sqrt{n}}|B| = \frac{(n-1)(n-2)}{2}|H|$. Since b_{11} and H have different signs, $b_{11} = -\frac{(n-1)(n-2)}{2}H$.

Hence $a_{11} = -\frac{n(n-1)}{2}H$, $a_{ii} = \frac{n}{2}H, i = 2, ..., n$, and $a_{ij} = 0, i \neq j, i, j = 1, ..., n$, that is, A is a diagonal matrix with a_{ii} given above.

Conversely, A in (1) and (2) satisfy the equality in (3.3). The proof is complete.

□

Applying Proposition 3.1 to hypersurfaces, we obtain the following Proposition 3.2, which can be used to prove Theorem 3.1 and may have its independent interest.

Proposition 3.2. *Let N be an $(n+1)$-dimensional manifold and M be a hypersurface in N with mean curvature H (not necessarily constant). Then, for any local orthonormal frame $\{e_i\}, i = 1, ..., n,$ of M,*

$$|A|^2 + nHh_{11} - \sum_{i=1}^{n} h_{1i}^2 \geq \frac{n^2(5-n)H^2}{4}, \tag{3.4}$$

where the second fundamental form $A = (h_{ij}), h_{ij} = \langle Ae_i, e_j \rangle, i, j = 1, ..., n$. Furthermore, the equality in (3.4) holds for some $\{e_i\}$ at a point $p \in M$, if and only if one of the following cases occurs at p:

(1) $n = 2$, $A = HI$, where I is the identity map, that is, M is umbilic at p;

(2) $n \geq 3$, A is a diagonal matrix with $a_{11} = -\frac{n(n-1)}{2}H$, $a_{ii} = \frac{n}{2}H, i = 2, ..., n$, and $a_{ij} = 0, i \neq j, i, j = 1, ..., n$, that is, M has $n-1$ equal principle curvature and only one is different when $H \neq 0$ at p, or M is totally geodesic when $H = 0$ at p.

Schoen and Yau ([SY], cf. [LW]) proved an inequality on harmonic functions on stable minimal hypersurfaces, we generalize their inequality to the CMC H-hypersurfaces:

Lemma 3.2. *Let M be a complete hypersurface with constant mean curvature H in N^{n+1}. Suppose that u is a harmonic function defined on M. If φ is a compactly supported function $\varphi \in H_{1,2}(M)$ such that $\varphi|\nabla u|$ satisfies the stability inequality (1.1), then*

$$\begin{aligned} \int_M \varphi^2 |\nabla u|^2 \{ \frac{1}{n}|\Phi|^2 - \sqrt{\frac{n-1}{n}}(n-2)H|\Phi| + (2n-1)\, H^2 \\ + \tilde{\mathrm{Ric}}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + \tilde{\mathrm{Ric}}\,(\nu,\nu) - \tilde{\mathrm{K}}\left(\frac{\nabla u}{|\nabla u|}, \nu\right) \} \\ + \int_M \frac{1}{n-1}\varphi^2 |\nabla|\nabla u||^2 \leq \int_M |\nabla\varphi|^2 |\nabla u|^2. \end{aligned} \tag{3.5}$$

Proof. Recall the Bochner formula

$$\frac{1}{2}\Delta|\nabla u|^2 = \mathrm{Ric}(\nabla u, \nabla u) + |\nabla^2 u|^2, \tag{3.6}$$

the equality $$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla u|\Delta|\nabla u| + |\nabla|\nabla u||^2, \tag{3.7}$$

and the inequality (see [LW]): when u is harmonic function,

$$|\nabla^2 u|^2 \geq \frac{n}{n-1}|\nabla|\nabla u||^2. \tag{3.8}$$

By (3.6), (3.7) and (3.8), we have

$$|\nabla u|\Delta|\nabla u| \geq \mathrm{Ric}(\nabla u, \nabla u) + \frac{1}{n-1}|\nabla|\nabla u||^2.$$

Let φ be a locally Lipschitz function with compact support on M. Choose $f = \varphi|\nabla u|$ in the stability inequality (1.1). Then

$$\begin{aligned}
&\int_M (|\Phi|^2 + \tilde{\mathrm{Ric}}\,(\nu,\nu) + nH^2)\varphi^2|\nabla u|^2 \\
\leq &\int_M |\nabla\,(\varphi\,|\nabla u|)\,|^2 \\
= &\int_M |\nabla\varphi|^2|\nabla u|^2 - 2\langle\varphi\,(\nabla\,|\nabla u|)\,,|\nabla u|\,\nabla\varphi\rangle + \int \varphi^2\,|\nabla|\nabla u||^2 \\
= &\int_M |\nabla\varphi|^2|\nabla u|^2 - \int \varphi^2|\nabla u|\Delta|\nabla u| \\
\leq &\int_M |\nabla\varphi|^2|\nabla u|^2 - \frac{1}{n-1}\int_M \varphi^2|\nabla|\nabla u||^2 - \int_M \varphi^2\mathrm{Ric}(\nabla u, \nabla u).
\end{aligned} \tag{3.9}$$

For any point $p \in M$ and any unit vector $\eta \in T_pM$, we choose a local orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$ at p such that $e_1 = \eta$, we have, from Gauss equation:

$$K(e_i, e_j) - \tilde{K}(e_i, e_j) = h_{ii}h_{jj} - h_{ij}^2, \text{ for } i, j = 1, 2, \cdots, n, \tag{3.10}$$

$$\begin{aligned}
\mathrm{Ric}(\eta,\eta) &= \sum_{i=2}^n \tilde{K}(\eta, e_i) + h_{11}\sum_{i=2}^n h_{ii} - \sum_{i=2}^n h_{1j}^2 \\
&= \tilde{\mathrm{Ric}}\,(\eta,\eta) - \tilde{K}\,(\nu,\eta) + nHh_{11} - \sum_{i=1}^n h_{1j}^2.
\end{aligned} \tag{3.11}$$

Substituting $\eta = \frac{\nabla u}{|\nabla u|}$ into (3.11) and then substituting (3.11) into (3.9),

we obtain

$$
\begin{aligned}
&\int_M (|\Phi|^2 + \tilde{\mathrm{Ric}}\,(\nu,\nu) + nH^2)\varphi^2|\nabla u|^2 \\
\leq &\int_M |\nabla\varphi|^2|\nabla u|^2 - \int_M \frac{1}{n-1}\varphi^2|\nabla|\nabla u||^2 \\
&- \int_M \varphi^2\{\tilde{\mathrm{Ric}}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) - \tilde{K}\left(\nu, \frac{\nabla u}{|\nabla u|}\right) + nHh_{11} - \sum_{i=1}^{n} h_{1i}^2\}. \quad (3.12)
\end{aligned}
$$

By Lemma 3.1,

$$
\begin{aligned}
&\int_M (|\Phi|^2 + \tilde{\mathrm{Ric}}\,(\nu,\nu) + nH^2)\varphi^2|\nabla u|^2 \\
\leq &\int_M |\nabla\varphi|^2|\nabla u|^2 - \int_M \frac{1}{n-1}\varphi^2|\nabla|\nabla u||^2 \\
&+ \int_M \varphi^2|\nabla u|^2\{(n-2)H|\Phi|\sqrt{\frac{n-1}{n}} + \frac{n-1}{n}|\Phi|^2 - (n-1)H^2 \\
&- \tilde{\mathrm{Ric}}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + \tilde{K}\left(\nu, \frac{\nabla u}{|\nabla u|}\right)\}.
\end{aligned}
$$

Thus (3.5) holds. □

As mentioned in Section 1, in order to estimate the number of the ends of hypersurface M, we need to discuss the nonexistence of nonconstant bounded harmonic functions on M with finite Dirichlet integral. We obtain that

Theorem 3.1. *Let M be a complete noncompact weakly stable CMC H-hypersurface in N^{n+1} in a manifold N. If, for any $p \in M$,*

$$
\tilde{\mathrm{Ric}}\,(X,X) + \tilde{\mathrm{Ric}}\,(\nu,\nu) - \tilde{K}\,(X,\nu) \geq \frac{n^2(n-5)}{4}H^2, X \in T_pM, |X| = 1,
$$

then M does not admit nonconstant bounded harmonic functions with finite Dirichlet integral.

Proof. We prove the conclusion by contradiction. Suppose there exists a nonconstant bounded harmonic function u with finite Dirichlet integral on M. Then there exists some point $p \in M$ such that $|\nabla u|(p) \neq 0$. Hence, $\int_{B_p(a)} |\nabla u| > 0$, for all $a > 0$.

We claim that u must satisfy $\int_M |\nabla u| = \infty$.

By the boundness of u, $\int_{B_p(R)} |\nabla u|^2 = \int_{\partial B_p(R)} u\frac{\partial u}{\partial r} \le C \int_{\partial B_p(R)} |\nabla u|$, where C is a constant. Hence when $R > 1$,

$$0 < C_0 = \int_{B_p(1)} |\nabla u|^2 \le \int_{B_p(R)} |\nabla u|^2 \le C \int_{\partial B_p(R)} |\nabla u|,$$

that is $\int_{\partial B_p(R)} |\nabla u| \ge C_1 > 0$.

By co-area formula,

$$\int_{B_p(R)} |\nabla u| = \int_1^R dr \int_{\partial B_p(r)} |\nabla u| \ge C_1(R-1). \tag{3.13}$$

Letting $R \to \infty$, we have $\int_M |\nabla u| = \infty$ as claimed.

Take, for $R > a$,

$$\varphi_1(a,R) = \begin{cases} 1, & \text{on } \bar{B}_p(a), \\ \frac{a+R-x}{R}, & \text{on } B_p(a+R)\backslash B_p(a), \\ 0, & \text{on } M\backslash B_p(a+R). \end{cases} \tag{3.14}$$

and

$$\varphi_2(a,R) = \begin{cases} 0, & \text{on } B_p(a+R), \\ \frac{a+R-x}{R}, & \text{on } B_p(a+2R)\backslash B_p(a+R), \\ -1, & \text{on } B_p(a+2R+b)\backslash B_p(a+2R), \\ \frac{x-(a+3R+b)}{R}, & \text{on } B_p(a+3R+b)\backslash B_p(a+2R+b), \\ 0, & \text{on } M\backslash B_p(a+3R+b), \end{cases} \tag{3.15}$$

where constant $b > 0$ will be determined later.

For any $\epsilon > 0$ given, we may choose large R such that $\frac{1}{R^2}\int_M |\nabla u|^2 < \epsilon$.

Define $\psi(t,a,R) = \varphi_1(a,R) + t\varphi_2(a,R)$, $t \in [0,1]$. We have

$$\int_M \psi(0,a,R)|\nabla u| \ge \int_{B_p(a)} |\nabla u| > 0,$$

and

$$\begin{aligned} \int_M \psi(1,a,R)|\nabla u| &= \int_M (\varphi_1(a,R) + \varphi_2(a,R))|\nabla u| \\ &\le \int_{B_p(a+R)} |\nabla u| - \int_{B_p(a+2R+b)\backslash B_p(a+2R)} |\nabla u|. \end{aligned} \tag{3.16}$$

By claim, for a and R fixed, we may find b sufficiently large, depending on a and R such that

$$\int_M \psi(1,a,R)|\nabla u| < 0.$$

By the continuity of $\psi(t,a,R)$ on t, there exists some $t_0 \in (0,1)$ depending on a and R such that $\int_M \psi(t_0,a,R)|\nabla u| = 0$.

Denote $\psi(t_0,a,R)$ by f. Since M is weakly stable, $f = \psi(t_0,a,R)|\nabla u|$ satisfies the stability inequality (1.2) and hence also satisfies Lemma 3.2.

Note the curvature condition implies that

$$\begin{aligned}
&\frac{1}{n}|\Phi|^2 - \sqrt{\frac{n-1}{n}}(n-2)H|\Phi| + (2n-1)\,H^2 \\
&\qquad + \tilde{\mathrm{Ric}}\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + \tilde{\mathrm{Ric}}\,(\nu,\nu) - \tilde{\mathrm{K}}\left(\frac{\nabla u}{|\nabla u|}, \nu\right) \\
&\qquad \geq \frac{1}{n}|\Phi|^2 - \sqrt{\frac{n-1}{n}}(n-2)H|\Phi| + \frac{(n-1)(n-2)^2}{4}H^2 \\
&\qquad \geq [\frac{|\Phi|}{\sqrt{n}} - \frac{\sqrt{n-1}(n-2)H}{2}]^2 \geq 0.
\end{aligned}$$

Then by (3.5), we have

$$\int_M \frac{1}{n-1} f^2 |\nabla|\nabla u||^2 \leq \int_M |\nabla f|^2 |\nabla u|^2. \tag{3.17}$$

Then

$$\begin{aligned}
&\frac{1}{n-1}\int_{B_p(a)} |\nabla|\nabla u||^2 \\
&\leq \int_{B_p(a+2R)\setminus B_p(a)} |\nabla\varphi_1|^2|\nabla u|^2 + t_0^2 \int_{B_p(a+3R+b)\setminus B_p(a+2R+b)} |\nabla\varphi_2|^2|\nabla u|^2 \\
&\leq \frac{1}{R^2}\int_{B_p(a+2R)\setminus B_p(a)} |\nabla u|^2 + \frac{1}{R^2}\int_{B_p(a+3R+b)\setminus B_p(a+2R+b)} |\nabla u|^2 \\
&\leq \frac{1}{R^2}\int_M |\nabla u|^2 < \epsilon.
\end{aligned}$$

In the above first inequality, we used $\langle \nabla\varphi_1, \nabla\varphi_2 \rangle = 0$.

By the arbitrariness of ϵ and a, $\nabla|\nabla u| \equiv 0$. So $|\nabla u| \equiv$ *constant*.

If $|\nabla u| \equiv$ *const.* $\neq 0$, then u is a nonconstant bounded harmonic function. This says M must be nonparabolic. Thus $\mathrm{vol}(M) = \infty$. Hence $\int_M |\nabla u|^2 =$

∞, which is impossible. Therefore $|\nabla u| \equiv 0$, $u \equiv$ *constant*. Contradiction. The proof is complete. $\square$

Remark 3.1. *If N is 3-dimensional, the curvature* $\tilde{\mathrm{Ric}}(X) + \tilde{\mathrm{Ric}}(Y) - \tilde{\mathrm{K}}(X,Y), X, Y \in T_pN, X \perp Y, |X| = |Y| = 1, p \in N$, *is equal to the scalar curvature* $\tilde{\mathrm{S}}$. *From the definition we know that the nonnegativity of the sectional curvature implies the nonnegativity of the above curvature. However, there are some examples showing that the converse may not be true (see [ShY]). In this paper, we adopt this curvature because it appears naturally in this context and provides more examples.*

Now we are ready to obtain the uniqueness of the ends of weakly stable CMC hypersurfaces. First, we consider the weakly stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ and obtain

Theorem 3.2. *(Th.0.3) If M is a complete oriented weakly stable minimal immersed hypersurface in $\mathbb{R}^{n+1}$, $n \geq 3$, then M must have only one end.*

Proof. First a complete minimal hypersurface in $\mathbb{R}^{n+1}$ must be compact. By Theorem 3.1, the dimension of the space $\mathcal{H}_D^0(M)$ is 1. By Proposition 2.3, each end of M must be nonparabolic. Hence by Theorem 1.1, M must have only one end. $\square$

Recall that Anderson ([An], Theorem 5.2) proved that a complete minimal hypersurface in $\mathbb{R}^{n+1}(n \geq 3)$ with finite total curvature and one end must be an affine-plane. Hence by the result of Anderson and Theorem 3.2, we have

Corollary 3.1. *(Cor.0.1) A complete weakly stable immersed minimal hypersurface M in $\mathbb{R}^{n+1}, n \geq 3$, with finite total curvature (i.e., $\int_M |A|^n < \infty$) is a hyperplane.*

Remark 3.2. *Y.B. Shen and X. Zhu [ShZ] showed that a complete stable immersed minimal hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature is a hyperplane. So Corollary 3.1 generalizes their result.*

Corollary 3.2. *A complete weakly stable CMC hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with finite total curvature (i.e., $\int_M |\Phi|^n < \infty$) is either a hyperplane or a geodesic sphere.*

Proof. do Carmo, Cheung and Santos [dCCS] proved that a complete stable CMC H-hypersurface, $H \neq 0$, in $\mathbb{R}^{n+1}, n \geq 3$ with finite total curvature must be compact. By their result, Theorem2.1 in [BC] and Corollary 3.1, we obtain that a complete weakly stable CMC hypersurfaces in $\mathbb{R}^{n+1}$ with finite total curvature must be a hyperplane or a geodesic sphere. $\square$

When the Ricci curvature of the ambient manifold has a strict low bound $-nH^2$, we obtain, for CMC H-hypersurfaces,

Theorem 3.3. *Let N^{n+1} be a complete Riemannian manifold of bounded geometry and M be a complete noncompact weakly stable immersed CMC H-hypersurface in N. If $\inf \tilde{Ric} > -nH^2$ and for any $p \in M$, $X \in T_pM, |X| = 1$,*

$$\tilde{\mathrm{Ric}}\,(X,X) + \tilde{\mathrm{Ric}}\,(\nu,\nu) - \tilde{K}\,(X,\nu) \geq \frac{n^2(n-5)}{4}H^2, \tag{3.18}$$

then M has only one end.

Proof. By Proposition 2.2, each end of M is nonparabolic. Hence, by Theorem 3.1 and Theorem 1.1, we get the conclusion. $\square$

Remark 3.3. *The curvature conditions in Theorem 3.3 demands, some positivity of curvature of N or the restriction on the dimension of N. See the evidence in its several consequences. However, it is worth to note that the same result holds without any curvature condition for minimal hypersurfaces in $\mathbb{R}^{n+1}, n \geq 3$, as Theorem 3.2 says. The reason is that we have a global Sobolev inequality (2.3) in this case.*

Now we derive some consequences of Theorem 3.3.

Corollary 3.3. *(Th.0.2) Let M be a complete noncompact weakly stable CMC H-hypersurface in the hyperbolic space $\mathbb{H}^{n+1}, n = 3, 4$. If $H^2 \geq \frac{10}{9}$, when $n = 3$; $H^2 \geq \frac{7}{4}$, when $n = 4$, respectively, then M has only one end.*

Proof. Observe that the hypotheses of Theorem 3.3 are satisfied. $\square$

In [Ch1] and [Ch2], Cheng proved that let N^{n+1} be an $(n+1)$-dimensional manifold and M be a complete CMC H-hypersurface immersed in N with finite index. Then M must be compact in the following two cases: 1) $n = 3,4,5$, $H = 0$, $\inf\{\tilde{\mathrm{Ric}}(w) + \tilde{\mathrm{Ric}}(\nu) - \tilde{\mathrm{K}}(w,\nu) |\ w \in T^1_pM,\ p \in M\} > 0$; 2) $n = 3,4$, $H \neq 0$, $\inf\{\tilde{\mathrm{Ric}}(w) + \tilde{\mathrm{Ric}}(\nu) - \tilde{\mathrm{K}}(w,\nu) |\ w \in T^1_pM,\ p \in M\} > \frac{n^2(n-5)}{4}H^2$. Combining this result, Theorem 3.2 and Theorem 3.3, we obtain that

Corollary 3.4. *If M is a complete noncompact weakly stable immersed CMC H-hypersurface in $\mathbb{R}^6$, then M has only one end.*

Proof. If $H \neq 0$, by Theorem 3.3, we know that when $n \leq 5$, M in $\mathbb{R}^{n+1}$ has only one end. But it is known ([Ch1]) that there is no complete noncompact weakly stable CMC H-hypersurfaces in $\mathbb{R}^4, \mathbb{R}^5$ ($H \neq 0$). Hence only the case $\mathbb{R}^6$ may occur. If $H = 0$, Theorem 3.2 says that a complete noncompact weakly stable minimal hypersurfaces in $\mathbb{R}^{n+1}, n \geq 3$, must have only one end. Combining two cases, we obtain the conclusion. □

The above result in [Ch1] and [Ch2] implies that any complete weakly stable H-hypersurface in a complete manifold N^{n+1}, $n = 3, 4$, or $n = 5$ and $H = 0$, of nonnegative sectional curvature must be compact. Hence by Theorem 3.3, we obtain the following

Theorem 3.4. *(Th.0.1) Let $N^{n+1}, n \geq 5$, be a complete Riemannian manifold and M be a complete noncompact weakly stable immersed CMC H-hypersurface in N. If one of the following cases occurs,*

(1) when $n = 5$, the sectional curvature of N is nonnegative and $H \neq 0$;

(2) when $n \geq 6$, the sectional curvature $\tilde{K}$ of N satisfies $\tilde{K} \geq \tau > 0$, and $H^2 \leq \frac{4(2n-1)}{n^2(n-5)}\tau$ for some number $\tau > 0$;

(3) when $n \geq 6$, the sectional curvature and the Ricci curvature of N satisfy $\tilde{K} \geq 0$, $\tilde{\mathrm{Ric}} \geq \tau > 0$, and $H^2 \leq \frac{4\tau}{n^2(n-5)}$, for some number $\tau > 0$,

then M has only one end.

In particular, any complete noncompact stable minimal hypersurface in a manifold N^{n+1}, $n \geq 6$, of nonnegative sectional curvature and Ricci curvature bounded from below by a positive number has only one end.

As some special cases, Theorem 3.4 implies that

Corollary 3.5. *A complete noncompact weakly stable CMC H-hypersurface has only one end, if it is in either*

1) the standard sphere $\mathbb{S}^6$ with $H \neq 0$; or

2) the standard sphere $\mathbb{S}^{n+1}, n \geq 6$, with $H^2 \leq \frac{4(2n-1)}{n^2(n-5)}$; or

3) $\mathbb{S}^k \times \mathbb{S}^l$, $k \geq 2, l \geq 2$, $k + l \geq 7$, with the product metric and $H = 0$.

In particular, a complete noncompact stable minimal hypersurface in $\mathbb{S}^{n+1}$, $n \geq 6$, and $\mathbb{S}^k \times \mathbb{S}^l$, $k \geq 2, l \geq 2, k + l \geq 7$, has only one end.

4 Property of Schrödinger operator on parabolic manifolds

In this section, we prove a property of Schrödinger operator for parabolic manifolds (not necessary a submanifold). It will be applied to weakly stable CMC hypersurfaces and also may have its independent interest.

Proposition 4.1. *Let M be a complete parabolic manifold with infinity volume. Consider the operator $L = \Delta + q(x)$ on M (here $q : M \to \mathbb{R}$ is a continuous function on M). If $q(x) \geq 0$ and $q(x)$ is not identically zero, then there exists a compactly supported piecewise smooth function ψ such that $\int_M \psi(x) = 0$ and $-\int_M \psi L\psi < 0$.*

Proof. By hypothesis, we may choose a point $p \in M$ such that $q(p) > 0$. Denote $C := \int_{B_p(1)} q(x)dv > 0$. Choose a monotonically increasing sequence $\{r_i\}$ with $r_i \to \infty$ and consider the harmonic functions g_i defined by

$$\begin{cases} \Delta g_i = 0, & \text{on } B_p(r_i)\backslash B_p(1), \\ g_i = 1, & \text{on } \partial B_p(1), \\ g_i = 0, & \text{on } \partial B_p(r_i). \end{cases}$$

Since M is parabolic, we have that $\lim_{r_i\to+\infty} \int_{B_p(r_i)\backslash B_p(1)} |\nabla g_i|^2 = 0$.

By this property, we can find some positive number $R_1 > 1$ and a corresponding function f_1 satisfying

$$\begin{cases} \Delta f_1 = 0, & \text{on } B_p(R_1)\backslash B_p(1), \\ f_1 = 1, & \text{on } \partial B_p(1), \\ f_1 = 0, & \text{on } \partial B_p(R_1), \end{cases}$$

and $\int_{B_p(R_1)\backslash B_p(1)} |\nabla f_1|^2 < \frac{C}{6}$.

Let

$$\varphi_1 = \begin{cases} 1, & \text{on } \bar{B}_p(1), \\ f_1, & \text{on } B_p(R_1)\backslash B_p(1), \\ 0, & \text{on } M\backslash B_p(R_1). \end{cases} \tag{4.1}$$

Similarly we can find a positive number $R_2 > R_1$ and a function f_2

satisfying $\int_{B_p(R_2)\backslash B_p(R_1)} |\nabla f_2|^2 < \frac{C}{6}$ and

$$\begin{cases} \Delta f_2 = 0, & \text{on } B_p(R_2)\backslash B_p(R_1), \\ f_2 = 1, & \text{on } \partial B_p(R_1), \\ f_2 = 0, & \text{on } \partial B_p(R_2). \end{cases}$$

Let

$$\varphi_2 = \begin{cases} 0, & \text{on } \bar{B}_p(R_1), \\ f_2 - 1, & \text{on } B_p(R_2)\backslash B_p(R_1), \\ -1, & \text{on } M\backslash B_p(R_2). \end{cases} \tag{4.2}$$

Again, for any constant $b > 0$, there exists $R_3 > R_2 + b$ and a function f_3 satisfying $\int_{B_p(R_3)\backslash B_p(R_2+b)} |\nabla f_3|^2 < \frac{C}{6}$ and

$$\begin{cases} \Delta f_3 = 0, & \text{on } B_p(R_3)\backslash B_p(R_2+b), \\ f_3 = -1, & \text{on } \partial B_p(R_2+b), \\ f_3 = 0, & \text{on } \partial B_p(R_3). \end{cases}$$

Let

$$\varphi_3 = \begin{cases} 0, & \text{on } \bar{B}_p(R_2+b), \\ f_3 + 1, & \text{on } B_p(R_3)\backslash B_p(R_2+b), \\ 1, & \text{on } M\backslash B_p(R_3). \end{cases} \tag{4.3}$$

Thus the sum of two functions $\varphi_2 + \varphi_3$ satisfies

$$\varphi_2 + \varphi_3 = \begin{cases} 0, & \text{on } B_p(R_1), \\ \varphi_2, & \text{on } B_p(R_2)\backslash B_p(R_1), \\ -1, & \text{on } B_p(R_2+b)\backslash B_p(R_2), \\ f_3, & \text{on } B_p(R_3)\backslash B_p(R_2+b), \\ 0, & \text{on } M\backslash B_p(R_3). \end{cases} \tag{4.4}$$

Let $\phi_t = \varphi_1 + t(\varphi_2 + \varphi_3)$. We see that ϕ_t has compact support in M. Then we define $\xi(t) := \int_M \varphi_1 + t\int_M(\varphi_2 + \varphi_3)$ we know that $\xi(0) = \int_M \varphi_1 > 0$ and since the volume of M is infinite we can choose b large such that

$$\begin{aligned} \xi(1) &= \int_M \varphi_1 + \int_M (\varphi_2 + \varphi_3) \\ &\leq \int_M \varphi_1 - \int_{B_p(R_2+b)\backslash B_p(R_2)} 1 < 0. \end{aligned}$$

So there exists a $t_0 \in (0,1)$ such that $\int_M \phi_{t_0} = 0$ and

$$\begin{aligned}
&-\int_M \phi_{t_0} L\phi_{t_0} = \int_M |\nabla\phi_{t_0}|^2 - q(x)\phi_{t_0}^2 \\
&\le \int_M |\nabla\varphi_1|^2 + \int_M |\nabla\varphi_2|^2 + \int_M |\nabla\varphi_3|^2 - \int_{B_p(1)} q(x)\varphi_1^2 \\
&= \int_{B_p(R_1)\backslash B_p(1)} |\nabla f_1|^2 + \int_{B_p(R_2)\backslash B_p(R_1)} |\nabla f_2|^2 \\
&\qquad\qquad + \int_{B_p(R_3)\backslash B_p(R_2+b)} |\nabla f_3|^2 - \int_{B_p(1)} q(x) \\
&< \frac{C}{6} + \frac{C}{6} + \frac{C}{6} - C = -\frac{C}{2} < 0.
\end{aligned}$$

Choosing ϕ_{t_0} as ψ, we obtain the conclusion of the proposition. □

Remark 4.1. *A special case of Proposition 4.1 that M is a surface was proved by da Silveira [dS] by using the conformal structure of ends of two-dimensional parabolic manifolds. This structure (obtained by using Huber's theorem) does not exist in higher dimensional cases.*

5 Structure of weakly stable CMC hypersurfaces

In this section, we will study the structure of a weakly stable CMC hypersurface according to its parabolicity or nonparabolicity.

(I) Parabolic case:

Applying Proposition 4.1 to the case that M is a weakly stable CMC hypersurface, we obtain

Proposition 5.1. *Let M be a complete weakly stable CMC H-hypersurface in N^{n+1}. Suppose that the Ricci curvature of N is bounded from below by $-nH^2$. If M is parabolic and has infinite volume, then M must be totally umbilic in N. Moreover the Ricci curvature $Ric(\nu,\nu)$ in the normal direction is identically equal to $-nH^2$ along M and the scalar curvature S_M is nonnegative.*

Proof. From the assumption, $|\Phi|^2 + \tilde{\mathrm{Ric}}(\nu,\nu) + nH^2 \ge 0$. Since M is weakly stable, by Proposition 4.1, it holds that $|\Phi|^2 + \tilde{\mathrm{Ric}}(\nu,\nu) + nH^2 \equiv 0$. Hence $\Phi \equiv 0$, that is, M is umbilic, and $\tilde{\mathrm{Ric}}(\nu,\nu) + nH^2 \equiv 0$.

At any point $p \in M$, choose a local orthonormal frame field $e_1, e_2, \cdots, e_n, \nu$ at p such that $e_1, e_2, \cdots, e_n$ are tangent fields.

Since $\Phi \equiv 0$, Gauss equations (3.10) become:

$$K(e_i, e_j) - \tilde{K}(e_i, e_j) = H^2, \text{ when } i \neq j. \tag{5.1}$$

Then

$$\sum_{i,j=1}^{n} K(e_i, e_j) - \sum_{i,j=1}^{n} \tilde{K}(e_i, e_j) - n(n-1)H^2 = 0, \tag{5.2}$$

$$\begin{aligned} S_M &= \sum_{i=1}^{n} [\tilde{\text{Ric}}(e_i, e_i) - \tilde{K}(\nu, e_i)] + n(n-1)H^2 \\ &\geq -n^2H^2 - \tilde{\text{Ric}}(\nu, \nu) + n(n-1)H^2 \\ &= 0. \end{aligned}$$

□

Theorem 5.1. *Let M be a complete weakly stable CMC hypersurface immersed in N^{n+1} with constant mean curvature H. Suppose that N has bounded geometry and the Ricci curvature of N is bounded from below by $-nH^2$. If M is parabolic, then M must be totally umbilic in N. Moreover the Ricci curvature $Ric(\nu, \nu)$ in the normal direction is identically equal to $-nH^2$ along M and the scalar curvature S_M is nonnegative.*

Proof. From Lemma 2.1 when N has bounded geometry, then the volume of M is infinite. Thus the conclusion follows directly from Proposition 5.1. □

Theorem 5.2. *(Th.0.4) Let N be a complete manifold of bounded geometry and M be a complete noncompact weakly stable CMC H-hypersurface immersed in N. If the sectional curvature of N is bounded from below by $-H^2$ and if M is parabolic, then it is totally umbilic and has nonnegative sectional curvature. Further, either*

(1) M has only one end; or

(2) $M = \mathbb{R} \times P$ with the product metric, where P is a compact manifold of nonnegative sectional curvature.

Proof. We have shown $\Phi \equiv 0$ in Theorem 5.1. Since $\tilde{K} \geq -H^2$, M has nonnegative (intrinsic) sectional curvature by the Gauss equation (5.1). If M has more than one end, by the splitting theorem of Cheeger and Gromoll [CG] on manifolds of nonnegative curvature, we get the conclusion (2). □

(II) Nonparabolic case:

In this situation, we apply Theorem 3.18 and obtain the following result:

Theorem 5.3. *(Th.0.5) Let N be a complete Riemannian manifold and M be a complete noncompact weakly stable H-hypersurface immersed in N. If M is nonparabolic, and*

$$\tilde{\mathrm{Ric}}(\nu) + \tilde{\mathrm{Ric}}(X) - \tilde{\mathrm{K}}(X, \nu) \geq \frac{n^2(n-5)}{4} H^2, \forall X \in T_pM, |X| = 1, p \in M,$$

then it has only one nonparabolic end.

Proof. Since M is nonparabolic, it has at least a nonparabolic end. If M has two or more nonparabolic ends, then the dimension of $\mathcal{H}_D^0(M)$ is not less than 2, which is a contradiction with Theorem 3.1.

□

When M is a weakly stable minimal hypersurface, combining Theorem 5.2 in (I) and Theorem 5.3 in (II), we obtain that,

Theorem 5.4. *Let N be a complete Riemannian manifold of bounded geometry and nonnegative sectional curvature and M be a complete noncompact oriented weakly stable minimal hypersurface immersed in N. Then*

(1) when M is parabolic, then either it has only one end and nonnegative curvature; or it is isometric to $\mathbb{R} \times P$ with the product metric, where P is a compact manifold of nonnegative curvature. Moreover M is totally geodesic;

(2) when M is nonparabolic, then it has only one nonparabolic end.

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